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Singularly perturbed differential equations with impulse action

ABSTRACT

of the PhD thesis for the degree of Doctor of Philosophy (PhD)
in the specialty «8D05401-Mathematics»

Mathematics studies mathematical models of natural processes. Every mathematical model is an approximation at some level. When constructing a mathematical model, one attempts to capture the most essential and primary features of the process under consideration. On the other hand, analytical or numerical calculations should be simple enough to provide the necessary information about the process.

The relevance of the study. The interest in singularly perturbed equations arises from the fact that they serve as mathematical models for diffusion, chemical kinetics, mathematical biology, physics, fluid dynamics, engineering, and many other applied problems. To give real-life examples, singular wave phenomena occur when a force is suddenly applied: an earthquake can generate a catastrophic tsunami, a rapid change in temperature may cause a thermal wave in a porous material, and the behavior of a singular nonlinear oscillator can change in highly nontrivial ways. Nowadays, singular wave motions have become an active research area in mathematics, particularly in the study of quasi-periodic bifurcations, singular dissipation, and related phenomena. The theory of such equations was established and developed in the works of L. Schlesinger, G. D. Birkhoff, P. Noayon, V. Vasov, A. H. Nayfeh, A. N. Tikhonov, M. I. Vishik, L. A. Lusternik, N. N. Bogolyubov, Yu. A. Mitropolsky, A. B. Vasiliev, V. F. Butuzov, R. E. O'Malley, D. R. Smith, W. Eckhaus, K. W. Chang, F. A. Howes, J. Kevorkian, J. D. Cole, J. Sanders, F. Verhulst, E. F. Mishchenko, N. Rozov, S. A. Lomov, M. I. Imanaliev, and K. A. Kasymov.

Consider the following model of a singularly perturbed differential equation

$$\varepsilon z' = f(z, y, t),$$

$$y' = g(z, y, t),$$

where ε is a small positive parameter. In the literature, results based on this system are referred to as Tikhonov-type theorems. Various asymptotic methods have been developed to approximate solutions of singularly perturbed problems. These methods make it possible to construct uniform approximations of arbitrary accuracy. One of the key approaches that emerged from this research is the boundary function method, also known as the boundary layer correction method. If the conditions of Tikhonov's theorem are satisfied in the domain under consideration, the boundary function method can be applied to solve singularly perturbed problems. In this thesis, the boundary function method is employed to obtain the main results.

Bainov and Kovachev first extended the impulse analogue of Tikhonov's theorem as follows

$$\varepsilon z' = f(z, y, t), \quad \Delta z|_{t=t_i} = I_i(y(t_i)), \tag{1a}$$

$$y' = g(z, y, t), \quad \Delta y|_{t=t_i} = J_i(y(t_i)), \tag{1b}$$

where $i = 1,2,...,p$ и $0 < t_1 < t_2 < ... < t_p < T$. The general impulse system consists of differential equations (1a) and impulse equations (1b). In previous articles and books, we considered systems in which the small parameter was present only in the differential equations but did not participate in the impulse part. That is, in these systems only the differential part was singularly perturbed.

Akhmet and Chag first considered the following system in the literature, in which in addition to the differential equation, the impulsive part is also singular:

$$\begin{aligned} &\varepsilon z' = f(z,y,t), \qquad y' = g(z,y,t), \\ &\varepsilon \Delta z\,|_{t=\theta_i} = I(z,y,\varepsilon), \;\; \Delta y\,|_{t=\eta_j} = J(z,y), \end{aligned} \tag{2}$$

where ε is a positive small parameter, z, f and $I - m$ are dimensional functions, y, g and $J - n$ are dimensional functions, $\theta_i, i = 1,2,...,p, \quad 0 < \theta_1 < \theta_2 < ... < \theta_p < T -$ discontinuous moments in the interval $(0,T)$, z^0 and $y^0 -$ are independent of ε.

In addition, the following conditions are set to prevent the impulse function from going to infinity if the small parameter of the impulse function tends to zero:

$$\lim_{(z,y,\varepsilon)\to(\varphi,\bar{y},0)} \frac{I(z,y,\varepsilon)}{\varepsilon} = 0 \text{ and } \lim_{(z,y,\varepsilon)\to(\varphi,\bar{y},0)} \frac{I(z,y,\varepsilon)}{\varepsilon} = I_0 \neq 0, \tag{*}$$

According to this condition, two cases are considered: single-layer singularity and multilayer singularity.

In this thesis, not only singular differential equations but also singularly perturbed impulsive systems are considered. The aim of the work is to construct a complete asymptotic series for the solution of a singularly perturbed impulsive system with arbitrary accuracy. In what follows, instead of the phrase *"we construct an asymptotic series"*, the expression *"we construct asymptotics"* is used. By this we mean an algorithm that makes it possible to determine any n-th term of the asymptotic series. For this purpose, the boundary function method is applied, and a uniform asymptotic approximation of the solution on the segment $0 \le t \le T$ is obtained.

Aim of the study. The aim of the dissertation is to construct the asymptotic expansion of the solution of the initial-value problem for singularly perturbed impulsive differential equations and to determine the singularities of the problem under consideration.

Object of study. The initial-value problem for singularly perturbed impulsive differential equations.

Research methods. Effective combinations of the following advanced methods are employed in the dissertation:

- Asymptotic methods for the integration of singularly perturbed differential equations;
- General theory of singularly perturbed differential equations;
- Boundary layer methods;
- Theory of impulsive differential equations.

Theoretical and practical significance. Differential equations with small parameters in higher derivatives have been extensively studied due to their wide range of applications in hydrodynamics (e.g., the Navier–Stokes equations), radio engineering, mechanics, physics, and engineering (e.g., the Schrödinger equation).

Singularly perturbed impulsive differential equations provide accurate models of many natural phenomena, such as controlling chaos and bifurcations in engineering systems, modeling epidemic outbreaks with impulsive effects, and analyzing as well as controlling complex dynamical systems. Because of their complexity, obtaining exact solutions is generally difficult, which explains the growing interest in this field.

Scientific novelty and main results proposed for defence. In this dissertation, problems in previously unexplored formulations are investigated, and new results are obtained and proposed for defence:

- An asymptotic expansion is constructed for the initial-value problem of singularly perturbed linear differential equations with impulses;
- An asymptotic expansion is constructed for the initial-value problem of singularly perturbed nonlinear differential equations with impulses;
- An algorithm is developed that determines the conditions for asymptotic approximation of impulsive differential equations with singular perturbations;
- Depending on the conditions imposed on the impulse function, single-layer and multi-layer singular phenomena are identified;
- A justification of the constructed asymptotics is provided;
- Numerical examples based on modeling are presented.

Approbation of the work. The results of the dissertation work were presented at the International Scientific Conference ‘Farabi World’ (Almaty, Kazakhstan, 6-8 April 2021), at the International Scientific Conference ‘Farabi World’ (Almaty, Kazakhstan, 6-7 April 2023), at the International Scientific and Practical Conference ‘Actual Problems of Mathematics, Physics and Information Technologies in Education’ (Osh, Kyrgyzstan, 26-27 September 2024).), at the International Scientific Conference ‘Ufa Autumn Mathematical School-2024’ (Ufa, Russia, 2-5 October 2024), “At the Traditional April Mathematical Conference dedicated to the 80th Anniversary of the Institute of Mathematics and Mathematical Modeling and timed to the Day of Science” (Almaty, Kazakhstan, April 1–4, 2025), as well as at the scientific seminar of the Department of Mathematics, Faculty of Mechanics and Mathematics, Al-Farabi Kazakh National University , where the presentation was made and discussed.

Scientific conclusions, their reliability and validity. The results obtained in the course of the study are supported by evidence previously presented by renowned scientists, publications in indexed international journals and conference proceedings.

Publications. Based on the results of the dissertation, 12 papers have been published:

- *Publications in the editions included in Web of Science and Scopus databases:*

1. Asymptotic solutions of differential equations with singular impulses // Carpathian Journal of Mathematics. –2024,–40, no. 3, 581 – 598.
2. A case of impulsive singularity // Journal of Mathematics, Mechanics and Computer Science, [S.l.], v. 117, n. 1, apr. 2023. ISSN 2617-4871
3. Asymptotic convergence of solutions for singularly perturbed linear impulsive systems with full singularity // Symmetry. 2025; 17(9):1389. Scopus: 93%, Web of Science: Q2.
4. Asymptotic behavior of the solution of the integral boundary value problem for singularly perturbed integro-differential equations // Journal of Mathematics,

Mechanics and Computer Science, [S.l.], v. 112, n. 4, dec. 2021. ISSN 2617-4871.

5. Asymptotic expansion of the solution for singular perturbed linear impulsive systems // Journal of Mathematics, Mechanics and Computer Science, –2024, –122(2), 14–26.

6. Asymptotic solutions to initial value problems for singularly perturbed quasi-linear impulsive systems// Journal of Mathematics, Mechanics and Computer Science. 2025; 127(3): 13–28. Scopus: 18%, Web of Science: Q4.

- *Publications in the proceedings of international scientific conferences:*

1. Asymptotic feature of the solution of the boundary value problem for singularly perturbed integral-differential equation // International Scientific Conference «FARABI ƏLEMI», Almaty, 7-8 April 2021: abstracts (Almaty - 2021. - 12 p.).

2. System of singularly perturbed impulse differential equations // International Scientific Conference «FARABI ƏLEMI», Almaty, 7-8 April 2023: abstracts (Almaty - 2023. - 5 p.).

3. Asymptotic solutions of differential equations with singular impulses // International Scientific and Practical Conference «Actual Problems of Mathematics, Physics and Information Technologies in Education», Osh, Kyrgyzstan, 26-27 September 2024: abstracts (Osh - 2024. - 17 p.).

4. Asymptotic decomposition of the solution of the problem for singularly perturbed linear impulsive systems with singular impulses // International Conference «Ufa Autumn Mathematical School-2024», Ufa, Russia, 2-5 October 2024: abstracts (Ufa - 2024. - 2 vol., 7-8 p.)

5. Asymptotic solutions of differential equations with singular impulses // Abstracts of the “Traditional April Mathematical Conference dedicated to the 80th Anniversary of the Institute of Mathematics and Mathematical Modeling on the Day of Science” (Almaty, Kazakhstan, April 1–4, 2025). – Almaty, 2025. – P. 82.

6. Asymptotic solutions of differential equations with singular impulses // 10th International Scientific Conference on Mathematics, Mechanics and Information Technologies, Kazakhstan, Almaty, September 24–26, 2025.

Based on the results of the dissertation research, 6 papers were published, including - 1 paper included in the first (Q1) quartile according to Clarivate Analytics Journal Citation Reports and/or with a CiteScore of 81 in the Scopus database, - 1 paper included in the first (Q2) quartile according to Clarivate Analytics Journal Citation Reports and/or with a CiteScore of 93 in the Scopus database and 4 papers in scientific journals included in the fourth (Q4) quartile with a percentile of 18; - 6 publications in collections of abstracts from international conferences.

Structure of the dissertation. The dissertation consists of normative references, an introduction, two main chapters (each subdivided into sections), a conclusion, and a list of references. The total length of the dissertation is 81 pages.

Content of the dissertation. The introduction presents the relevance and scientific novelty of the research topic, research methods, theoretical and practical significance, as well as a brief overview of the main results.

The **first chapter** is devoted to the asymptotic expansion of the solution of the initial-value problem for singularly perturbed impulsive linear differential equations.

An algorithm for determining the approximate terms of the solution is developed. Furthermore, an analogue of the Butuzov–Vasiliev theorem is formulated, and the existence and uniqueness of the solution are established.

In **Section 1.1**, the initial-value problem for singularly perturbed linear differential equations with variable coefficients is considered, and a single-layer singular phenomenon associated with the impulse function is defined.

In **Section 1.2**, the initial-value problem for singularly perturbed linear differential equations with variable coefficients is considered, and multilayers singular phenomenon associated with the impulse function is defined.

In **Section 1.3**, the initial-value problem for impulsive differential equations with singularly perturbed linear coefficients is analyzed, and multilayer singular phenomena associated with the impulse function are established.

The **second chapter** examines the initial-value problem for impulsive differential equations with singularly perturbed nonlinear coefficients. Both single-layer and multilayer singular phenomena related to the impulse function are identified. Numerical examples obtained through simulations are presented to confirm the theoretical results.

The **conclusion** summarizes the main findings of the dissertation and outlines the principal results achieved in the research.